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AXISYMMETRIC WAVES ON THE SURFACE OF VISCOUS FLUID

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An approximate solution is obtained for the linearized system of Navier — Stokes equations at low Reynolds numbers with boundary conditions corresponding to the case when axisymmetric normal and radial tangential stresses act at the surface of a heavy, viscous incompressible fluid. The axisymmetric shape of the free surface is specified at the initial instant of time. An integral definition of its shape valid for considerable times is obtained for stationary perturbations at the surface. Existence of circular waves which not only propagate from the perturbation source but, also, towards it, is established. Examples are considered. Waves propagating from the perturbation source are also investigated. It is established that the main part of such waves is the same for low and high (see, e.g., [1 — 3]) Reynolds numbers.

1. Let us assume that a heavy incompressible viscous fluid up to the instant of time $t' = 0$ is at reset in the half-space $z' \leqslant 0$ (the z'-axis is directed counter the force of gravity). Let at $t' \geqslant 0$ the tangential τ' and normal f' stresses begin to act on the free surface, and that the shape of the surface at instant $t' = 0$ is defined in the stationary cylindrical system of coorinates (r', θ, z'), whose origin is on the unperturbed free surface $z' = 0$, by an equation of the form $z' = h'(r')$. Assuming that all perturbations are axisymmetric and that the projection of the tangential stress on the transverse axis is zero. Then $\tau' = \tau'(r', t')$ and $f' = f'(r', t')$. At low Reynolds numbers the axisymmetric problem of the motion of fluid induced by small perturbations on its surface consists of the determination of velocity $\mathbf{v}' = \{v_r', v_z'\}$ and hydrodynamic pressure p' as functions of r', z', t' of the linearized Navier — Stokes equations

$$\frac{\partial \mathbf{v}'}{\partial t'} + \nabla q' = \nu \nabla^2 \mathbf{v}', \quad \nabla \mathbf{v}' = 0, \quad q' = \frac{p'}{\rho} + gz' \tag{1.1}$$

for specified on the unknown free surface $z' = \zeta'(r', t')$ of normal and tangential components of the stress tensor

$$\rho g \zeta' - \rho q' + 2\mu \frac{\partial v_z'}{\partial z'}\bigg|_{z'=0} = -f'(r', t'), \quad \mu \left(\frac{\partial v_z'}{\partial r'} + \frac{\partial v_r'}{\partial z'}\right)_{z'=0} = -\tau'(r', t') \tag{1.2}$$

(in a linear formulation conditions (1.2) are specified at the unperturbed surface $z' = 0$) and under condition that the unknown functions vanish as $z' \to -\infty$. At the initial instant of time

$$t' = 0, \quad \mathbf{v}' = 0, \quad \zeta' = h'(r') \tag{1.3}$$

function ζ' satisfies the equation

$$\partial\zeta'/\partial t' = v_z'\,|_{z'=0} \tag{1.4}$$

We introduce the dimensionless variables

$$R = \frac{gL^3}{\nu^2}, \quad r = r'\sqrt[3]{\frac{g}{\nu^2 R}}, \quad t = t'\sqrt[3]{\frac{g^2}{\nu R^2}}, \quad v = \frac{\nu v'}{gL^2}, \quad h = \frac{h'}{RL}$$

$$\zeta = \frac{\zeta'}{RL}, \quad z = \frac{z'}{L}, \quad q = \frac{q'}{gL}, \quad f = \frac{f'}{\rho g L}, \quad \tau = \frac{\tau'}{\rho g L}$$

where L is a characteristic length determined by the explicit form of surface perturbations.

The substitution of variables

$$v_r = \frac{\partial\varphi}{\partial r} - \frac{\partial w}{\partial z}, \quad v_z = \frac{\partial\varphi}{\partial z} + \frac{1}{r}\frac{\partial(rw)}{\partial r}, \quad q = -\frac{\partial\varphi}{\partial t}$$

reduces problem (1.1) — (1.4) to the dimensionless form

$$\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = 0 \tag{1.5}$$

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2} + \frac{\partial^2 w}{\partial z^2} \tag{1.6}$$

$$\frac{\partial\zeta}{\partial t} = \frac{\partial\varphi}{\partial z} + \frac{1}{r}\frac{\partial(zw)}{\partial r}\Big|_{z=0} \tag{1.7}$$

$$\frac{\partial\varphi}{\partial t} - 2\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}\right) + \frac{2}{r}\frac{\partial^2(rw)}{\partial r\partial z}\Big|_{z=0} + R\zeta = -f(r,\ t) \tag{1.8}$$

$$\frac{\partial w}{\partial t} - 2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2}\right) - 2\frac{\partial^2\varphi}{\partial r\partial z}\Big|_{z=0} = \tau(r,\ t) \tag{1.9}$$

$$t = 0, \quad \zeta = h(r), \quad \varphi = w = 0$$

We apply to these formulas the Laplace — Carson transformation with respect to t, then we apply the zero order Hankel transformation to (1.5), (1.7), and (1.8) and the first order Hankel transformation with respect to r to (1.6) and (1.9). To denote Hankel representations we use subscripts equal to the transformation order. As the result we obtain for the representations a system of ordinary differential equations and boundary conditions with constant coefficients. This system is used for determining the representations in the class of functions that vanish as $z \to -\infty$. Passing to the originals we obtain formulas for the free surface shape. For considerable times t and stationary perturbations at the surface when $t \geqslant 0$ $f = f(r)$, and $\tau = \tau(r)$, the free surface shape is defined by

$$\zeta(r,\ t) = \zeta_h(r,\ t) + \zeta_n(r,\ t) + \zeta_\tau(r,\ t) \tag{1.10}$$

$$\zeta_h = \int_0^\infty s h_0(s) H(t,\ s) J_0(rs)\,ds, \quad \zeta_n = -\int_0^\infty s f_0(s)\chi(t,\ s) J_0(rs)\,ds$$

$$\zeta_\tau = \int_0^\infty s\tau_1(s)\Psi(t,\ s)\,J_0(rs)\,ds$$

$$(f_0,\ h_0) = \int_0^\infty r\,[f(r),\ h(r)]\,J_0(rs)\,dr, \quad \tau_1 = \int_0^\infty r\tau(r)\,J_1(rs)\,dr$$

$$H(t, s) \leftarrow\!\!\div \frac{\Delta - Rs}{\Delta}, \quad \Psi(t, s) \leftarrow\!\!\div \frac{s(\sigma + 2s^2 - 2s\sqrt{\sigma + s^2})}{\sigma\Delta}$$

$$\Delta = (\sigma + 2s^2)^2 - 4s^3\sqrt{\sigma + s^2} + Rs$$

$$\chi(t, s) = \int_0^\infty \omega(\tau, s)d\tau - \int_t^\infty \omega(\tau, s)\,d\tau \tag{1.11}$$

$$\omega(t, s) \leftarrow\!\!\div \Omega(\sigma, s) = \frac{\sigma s}{\Delta} = s\sum_{k=1}^{4} \frac{1}{F_q'(q_k, s)} \frac{\sigma}{\sqrt{\sigma + s^2} - sq_k}$$

where σ is the parameter of the Laplace — Carson transformation, and $q_k = q_k(s)$ are roots of the polynomial

$$F(q, s) = s^3(q^4 + 2q^2 - 4q + 1) + R \tag{1.12}$$

In the half-plane $\operatorname{Re} q > 0$ with $0 < s < a \approx 1.2R^{1/4}$ polynomial (1.12) has two complex-conjugate roots $q_2 = \bar{q}_1$, and when $a < s < \infty$ it has two real roots: $0.68 < q_1 < 1$ and $0.30 < q_2 < 0.63$. It follows from (1.12) that

$$\operatorname{Re} q_k^2 < 1, \quad 0 \leqslant s < \infty, \quad k = 1,2,3,4 \tag{1.13}$$

2. Since (1.13) implies the integrability of $\omega(t, s)$ for $0 \leqslant t \leqslant \infty$, hence from (1.11) we have

$$\int_0^\infty \omega(t, s)\,dt = \lim_{\sigma\to 0} \frac{\Omega(\sigma, s)}{\sigma} = \frac{1}{R}$$

Using the integral formula of Fourier — Bessel, from (1.10) and (1.11) we obtain

$$\zeta_n = -\frac{f(r)}{R} + \int_0^\infty \int_t^\infty sf_0(s)\,\omega(\tau, s)\,J_0(rs)\,d\tau ds \tag{2.1}$$

From the second of formulas (1.11) follows that

$$\sum_{k=1}^{4} \frac{1}{F_q'(q_k, s)} = 0 \tag{2.2}$$

The explicit expression for $\omega(t, s)$ is now obtained from the second of formulas (1.11) and from (2.2), with allowance for the conversion defined in [4], in the form

$$\omega(t, s) = s^2 \sum_{k=1}^{4} \frac{q_k \operatorname{erfc}(-sq_k\sqrt{t}) \exp[s^2(q_k^2 - 1)t]}{F_q'(q_k, s)} \tag{2.3}$$

Using in the case of considerable times t the asymptotic expressions for the error function [5] and taking into account (2.2) and (1.12), for (2.3) we obtain formula

$$\omega(t, s) = \frac{1}{2s} \sum_{\operatorname{Re} q_k > 0} \frac{q_k \exp[s^2(q_k^2 - 1)t]}{q_k^3 + q_k - 1} + O(t^{-3/2}e^{-s^2 t}) \tag{2.4}$$

From (2. 1) and (2. 4) we have

$$t\to\infty,\quad \zeta_n = -\frac{f(r)}{R} + \zeta_n^{(1)}(r,t) + \zeta_n^{(2)}(r,t) \tag{2.5}$$

$$\zeta_n^{(1)} = -\operatorname{Re}\int_0^a \frac{q_1 \exp[s^2(q_1^2-1)t]}{s^2(q_1^2-1)(q_1^3+q_1-1)} f_0(s)J_0(rs)\,ds$$

$$\zeta_n^{(2)} = -\frac{1}{2}\int_a^\infty \frac{q_1 \exp[s^2(q_1^2-1)t]}{s^2(q_1^2-1)(q_1^3+q_1-1)} f_0(s)J_0(rs)\,ds$$

From (1. 12) we obtain

$$q_1 = s^{-3/4}R^{1/4}e^{i\pi/4} - {}^1/_2 s^{3/4}R^{-1/4}e^{-i\pi/4} + O(s),\quad s\to 0 \tag{2.6}$$

$$q_1 = 1 - {}^1/_4 Rs^{-3}[1+O(s^{-1})],\quad s\to 0$$

This with (1. 13) shows that the main contribution to the asymptotic expansion of integrals (2. 5) as $t\to\infty$ is provided by the neighborhoods of zero values of exponent indices. In the first of integrals (2. 5) that neighborhood coincides with that of point $s=0$. The related contribution is obtained by substituting a small segment $0\leqslant s\leqslant\varepsilon$ for the integration interval, taking into account that in that region by virtue of (2. 6)

$$s^2(q_1^2-1)\sim -2s^2 + i\sqrt{Rs},\quad \frac{q_1}{s^2(q_1^2-1)(q_1^3+q_1-1)}\sim -\frac{s}{R},\ s\to 0$$

and integrating from 0 to ∞. We have

$$\zeta_n^{(1)} = \frac{1}{R}\int_0^\infty s e^{-2s^2t} f_0(s)J_0(rs)\cos(t\sqrt{Rs})\,ds,\quad t\to\infty \tag{2.7}$$

The main contribution to the asymptotic expansion of the second of integrals (2. 5) as $t\to\infty$ is provided by the neighborhood of an infinitely distant point. By virtue of (2. 6) we have

$$s^2(q_1^2-1)\sim -\frac{R}{2s},\quad \frac{q_1}{s^2(q_1^2-1)(q_1^3+q_1-1)}\sim -\frac{2s}{R},\quad s\to\infty$$

Hence

$$\zeta_n^{(2)} = \frac{1}{R}\int_0^\infty s\exp\left[\frac{-Rt}{(2s)}\right] f_0(s)J_0(rs)\,ds,\ t\to\infty \tag{2.8}$$

Formulas (2. 5), (2. 7) and (2. 8) define the asymptotic behavior of perturbations on the surface, induced by stationary normal pressures. From (1. 10) we similarly obtain the integral representations of the free surface shape due to it initial rise. It is of the form

$$\zeta_h = \zeta_h^{(1)}(r,t) + \zeta_h^{(2)}(r,t),\quad t\to\infty \tag{2.9}$$

$$\zeta_h^{(1)} = \int_0^\infty s e^{-2s^2t} h_0(s)J_0(rs)\cos(t\sqrt{Rs})\,ds$$

$$\zeta_h^{(2)} = \int_0^\infty s \exp\left[\frac{-Rt}{2s}\right] h_0(s) J_0(rs)\, ds$$

Under the action of stationary radial tangential stresses we have

$$\zeta_\tau = \zeta_\tau^{(1)}(r, t) + \zeta_\tau^{(2)}(r, t), \quad t \to \infty \tag{2.10}$$

$$\zeta_\tau^{(1)} = -\frac{1}{R}\int_0^\infty s e^{-2s^2 t} \tau_1(s) J_0(rs) \cos(t\sqrt{Rs})\, ds$$

$$\zeta_\tau^{(2)} = \frac{1}{8}\int_0^\infty \frac{\tau_1(s)}{s^2} \exp\left[\frac{-Rt}{2s}\right] J_0(rs)\, ds$$

3. We shall now consider several specific examples.

1°. The asymptotic behavior of the surface of fluid from which a cylinder of radius a' initially submerged to a depth h' is removed at $t' = 0$.

The initial shape of the free surface is defined in dimensionless variables by

$$0 \leqslant r \leqslant a, \quad h(r) = -h; \quad r > a, \ h(r) = 0 \tag{3.1}$$

$$V' = \pi a'^2 h', \quad L = V'^{1/3}, \quad a = a'(g\nu^{-2}R^{-1})^{1/3}$$

$$h = h' / (RL), \quad R = g\nu^{-2}V'$$

where V' is the volume of fluid displaced by the cylinder at the initial instant of time. From (3.1) and (1.10) we obtain

$$h_0(s) = -ahJ_1(as) / s \tag{3.2}$$

Formula (2.9) then assumes the form

$$\zeta_h^{(2)} = -ah\int_0^\infty \exp\left[-\frac{Rt}{2s}\right] J_1(as) J_0(rs)\, ds \tag{3.3}$$

Let us consider the asymptotic behavior of the fluid surface when a and r are large. The neighborhood of point $s = 0$ is in this case immaterial. Hence, using asymptotic formulas for Bessel functions for large values of the argument, we obtain

$$\zeta_h^{(2)} = -2\pi^{-1}ha^{1/2}r^{-1/2}\{\operatorname{sgn}(r-a) \operatorname{Im} K_0[e^{i\pi/4}(2Rt|r-a|)^{1/2}] - \operatorname{Re} K_0[e^{i\pi/4}(2Rt(r+a))^{1/2}]\} \tag{3.4}$$

where K_0 is the Macdonald function.

When $r \gg a$, using the asymptotic formulas for the Macdonald functions, we have

$$\zeta_h^{(2)} = h[2a^2 / (\pi^2 r^2 \rho^*)]^{1/4}\, \xi(r, t) \tag{3.5}$$

$$\xi(r, t) = \exp(-\rho^{*1/2}) \sin(\pi/8 + \rho^{*1/2}) \operatorname{sgn}(r-a)$$

$$\rho^* = R\,|r-a|\,t$$

In the neighborhood of the circle $r = a$ formula (3.4) assumes the form

$$\zeta_h^{(2)} = -2h\pi^{-1} \operatorname{sgn}(r-a) \operatorname{Im} K_0(e^{i\pi/4}\sqrt{2}\rho^*)$$

which shows that damping of fluid oscillations occurs also in the neighborhood of $r = a$. For $2R \mid r - a \mid t \ll 1$ and small z we obtain

$$K_0(z) \sim \ln (2 / z), \quad \zeta_h^{(2)} \sim {}^1/_2 h \operatorname{sgn} (r - a)$$

Consequently in this case the free surface height at transition over the circle $r = a$ changes abruptly from $(- h / 2)$ to $h / 2$.

For the neighborhood of $r = 0$ formula (3.3) yields

$$\zeta_h^{(2)} = - ah \int_0^\infty \exp \left[- \frac{Rt}{2s} \right] J_1 (as)\, ds$$

Substituting here the asymptotics for considerable values of the argument for J_1 we obtain

$$\zeta_h^{(2)} = -h \sqrt{2} \exp (- \sqrt{aRt}) \cos \sqrt{aRt}$$

It follows from (3.5) that in this case perturbations $\zeta_h^{(2)}$ are waves propagating at velocity $(a' - r') / t'$ toward the source, i.e. to the circle $r' = a'$. In the course of time the perturbations are damped at any r'. The slowest damping occurs in the neighborhood of the perturbations source $r' = a'$.

Waves $\zeta_h^{(2)}$ and $\zeta_\tau^{(2)}$ represent similar perturbations (see (2.8) and (2.10) which also propagate toward their sources. The possibility of existence of such waves is revealed here for the first time.

In the investigation of perturbations (2.9) we set $a \to 0$ and $h \to \infty$ so that $V = \pi a^2 h =$ const. Formula (3.2) assumes the form $h_0 (s) = -V / (2\pi)$, and consequently

$$\zeta_h^{(1)} = - \frac{V}{2\pi} \int_0^\infty s \exp (- 2s^2 t) J_0 (rs) \cos (t \sqrt{Rs})\, ds \tag{3.6}$$

If $r \sim 0$ then $J_0 (rs) \sim 1$, and

$$\zeta_h^{(1)} = - \frac{V}{\pi t} \int_0^\infty x^3 e^{-2x^4} \cos \omega x\, dx = - \frac{6V2^{1/4}}{\pi t \omega^4} [1 + O (\omega^{-4})] \tag{3.7}$$

$$\omega = g^{1/2} t'^{3/4} \nu^{-1/4}$$

which shows that near the coordinate origin perturbations $\zeta_h^{(1)}$ are not of an oscillatory character.

Substituting for the Bessel function the respective asymptotics with $r \to \infty$, we obtain

$$t \to \infty, \quad \zeta_h^{(1)} = -2^{-1/2} \pi^{-1/2} V t^{-1/4} r^{-1/2} \operatorname{Re} (j_1 + j_2) \tag{3.8}$$

$$j_k = \int_0^\infty x^2 \exp (- 2x^4) \exp \{ i [\pm \omega (\eta x^2 \mp x) + \pi/4] \}\, dx$$

$$\eta = r' \nu^{-1/4} g^{-1/2} t'^{-5/4}; \quad k = 1, 2$$

where the plus sign corresponds to $k = 1$ and the minus sign to $k = 2$.

The asymptotic formula for j_1 with $\omega \to \infty$ is obtained by the method of stationary phase [6] in the form

$$r \to \infty, \quad \eta \ll \omega, \quad j_1 \sim -\frac{V \exp[-1/(8\eta^4)] \cos[\omega/(4\eta)]}{2^{5/2}\pi\eta^{5/2}\omega^{1/2}r^{1/2}t^{5/4}} \tag{3.9}$$

The function in the index of the oscillating exponent in j_2 has no stationary points for $x > 0$. Hence the estimate of j_2 assumes the form

$$j_2 = O(\omega^{-3}r^{-1/2}t^{-3/4}) \tag{3.10}$$

It follows from (3.8) and (3.9) that when r and t are independent, j_1 vanishes exponentially as $\omega \to \infty$ and j_2 predominates in (3.8). If, however, r and t are linked by the condition $\eta(r, t) = \text{const}$, then $j_1 = O(\omega^{-1/2}r^{-5/2}t^{-3/4})$, and j_1 exceeds the value [of j_2 in] (3.10). Thus perturbations $\zeta_h^{(1)}$ virtually vanish everywhere, except in region $\eta = \text{const}$, where they conform to (3.9). In dimensional form, taking into account (3.7) and (3.8), we have

$$\zeta_h'^{(1)} \sim -\frac{gt'^2V'\sqrt{2}}{8\pi r'^3} \exp\left(-\frac{\nu t'^5 g^2}{8r'^4}\right) \cos\left(\frac{gt'^2}{4r'}\right) \tag{3.11}$$

The condition $\eta = \text{const}$ means that circular waves (3.11) propagate from the perturbation source at velocity $5r' / 4t'$.

2°. The model of anticyclone at the fluid surface. In this case we assume the tangential stresses to be defined by $\tau'(r') = T'\delta(r' - a')$, where T' is the work of the radial tangential stresses on the surface per unit area of their propagation.

Let us assume that $T' \to \infty$ and $a' \to 0$ but the work $A' = \pi a'^2 T' = \text{const}$. From formula (2.10) we then obtain

$$\zeta_\tau'^{(1)} \sim -\frac{gA't'^4\sqrt{2}}{32\rho r'^5} \exp\left(-\frac{\nu g^2 t'^5}{8r'^4}\right) \cos\left(\frac{gt'^2}{4r'}\right)$$

3°. $f'(r') = f'$ for $0 \leqslant r' \leqslant a'$ and $f'(r') = 0$ for $r' > a'$.

If $a' \to 0$ and $f' \to \infty$, but $P' = \pi a'^2 f' = \text{const}$, we have a concentrated force acting on the free surface along the normal to it. In that case from (2.7) we have

$$\zeta_n'^{(1)} \sim \frac{P't'^2\sqrt{2}}{8\pi\rho r'^3} \exp\left(-\frac{\nu g^2 t'^5}{8r'^4}\right) \cos\left(\frac{gt'^2}{4r'}\right)$$

We note in conclusion that formula (3.11) coincides within $O(R)$ with that obtained in [1] by analyzing the exact solution of problem (1.1) – (1.4) for small values of parameter $\varepsilon = \nu g^{-1/2}L^{-3/2} = R^{-1/2}$. This means that the main part of the wave propagating from the perturbation source is the same for small and considerable values of R. In other words, the integral representations (2.7), (2.9) and (2.10), as well as the corresponding results obtained in [1] are valid for small and large values of parameter $R = \varepsilon^{-2}$.

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